NAG Toolbox for MATLAB

f08wa

1 Purpose

f08wa computes for a pair of n by n real nonsymmetric matrices (A, B) the generalized eigenvalues and, optionally, the left and/or right generalized eigenvectors using the QZ algorithm.

2 Syntax

```
[a, b, alphar, alphai, beta, vl, vr, info] = f08wa(jobvl, jobvr, a, b,
'n', n)
```

3 Description

A generalized eigenvalue for a pair of matrices (A, B) is a scalar λ or a ratio $\alpha/\beta = \lambda$, such that $A - \lambda B$ is singular. It is usually represented as the pair (α, β), as there is a reasonable interpretation for $\beta = 0$, and even for both being zero.

The right eigenvector v_j corresponding to the eigenvalue λ_j of (A, B) satisfies

$$Av_j = \lambda_j Bv_j.$$

The left eigenvector u_j corresponding to the eigenvalue λ_j of (A, B) satisfies

$$u_j^{\mathrm{H}} A = \lambda_j u_j^{\mathrm{H}} B.$$

where u_j^{H} is the conjugate-transpose of u_j.

All the eigenvalues and, if required, all the eigenvectors of the generalized eigenproblem $Ax = \lambda Bx$, where A and B are real, square matrices, are determined using the QZ algorithm. The QZ algorithm consists of four stages:

(i) A is reduced to upper Hessenberg form and at the same time B is reduced to upper triangular form.

(ii) A is further reduced to quasi-triangular form while the triangular form of B is maintained. This is the real generalized Schur form of the pair (A, B).

(iii) The quasi-triangular form of A is reduced to triangular form and the eigenvalues extracted. This function does not actually produce the eigenvalues λ_j, but instead returns α_j and β_j such that

$$\lambda_j = \alpha_j/\beta_j, \qquad j = 1, 2, \dots, n.$$

The division by β_j becomes your responsibility, since β_j may be zero, indicating an infinite eigenvalue. Pairs of complex eigenvalues occur with α_j/β_j and α_{j+1}/β_{j+1} complex conjugates, even though α_j and α_{j+1} are not conjugate.

(iv) If the eigenvectors are required they are obtained from the triangular matrices and then transformed back into the original co-ordinate system.

4 References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D 1999 *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphia URL: http://www.netlib.org/lapack/lug

Golub G H and Van Loan C F 1996 *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

Wilkinson J H 1979 Kronecker's canonical form and the QZ algorithm *Linear Algebra Appl.* **28** 285–303

5 Parameters

5.1 Compulsory Input Parameters

1: **jobvl – string**

If **jobvl** = 'N', do not compute the left generalized eigenvectors.

If **jobvl** = 'V', compute the left generalized eigenvectors.

Constraint: **jobvl** = 'N' or 'V'.

2: **jobvr – string**

If **jobvr** = 'N', do not compute the right generalized eigenvectors.

If **jobvr** = 'V', compute the right generalized eigenvectors.

Constraint: **jobvr** = 'N' or 'V'.

3: **a(lda,$*$) – double array**

The first dimension of the array **a** must be at least $\max(1,\mathbf{n})$

The second dimension of the array must be at least $\max(1,\mathbf{n})$

The matrix A in the pair (A,B).

4: **b(ldb,$*$) – double array**

The first dimension of the array **b** must be at least $\max(1,\mathbf{n})$

The second dimension of the array must be at least $\max(1,\mathbf{n})$

The matrix B in the pair (A,B).

5.2 Optional Input Parameters

1: **n – int32 scalar**

Default: The first dimension of the arrays **a**, **b** and the second dimension of the arrays **a**, **b**. (An error is raised if these dimensions are not equal.)

n, the order of the matrices A and B.

Constraint: $\mathbf{n} \geq 0$.

5.3 Input Parameters Omitted from the MATLAB Interface

lda, ldb, ldvl, ldvr, work, lwork

5.4 Output Parameters

1: **a(lda,$*$) – double array**

The first dimension of the array **a** must be at least $\max(1,\mathbf{n})$

The second dimension of the array must be at least $\max(1,\mathbf{n})$

a has been overwritten.

2: **b(ldb,$*$) – double array**

The first dimension of the array **b** must be at least $\max(1,\mathbf{n})$

The second dimension of the array must be at least $\max(1,\mathbf{n})$

b has been overwritten.

3: **alphar**($*$) – **double array**

Note: the dimension of the array **alphar** must be at least $\max(1, \mathbf{n})$.

The element **alphar**(j) contains the real part of α_j.

4: **alphai**($*$) – **double array**

Note: the dimension of the array **alphai** must be at least $\max(1, \mathbf{n})$.

The element **alphai**(j) contains the imaginary part of α_j.

5: **beta**($*$) – **double array**

Note: the dimension of the array **beta** must be at least $\max(1, \mathbf{n})$.

($\mathbf{alphar}(j) + \mathbf{alphai}(j) \times i)/\mathbf{beta}(j)$, for $j = 1, \ldots, \mathbf{n}$, will be the generalized eigenvalues.

If **alphai**(j) is zero, then the jth eigenvalue is real; if positive, then the jth and $(j+1)$st eigenvalues are a complex conjugate pair, with **alphai**($j+1$) negative.

Note: the quotients $\mathbf{alphar}(j)/\mathbf{beta}(j)$ and $\mathbf{alphai}(j)/\mathbf{beta}(j)$ may easily overflow or underflow, and **beta**(j) may even be zero. Thus, you should avoid naively computing the ratio α_j/β_j. However, $\max|\alpha_j|$ will always be less than and usually comparable with $\|\mathbf{a}\|_2$ in magnitude, and $\max|\beta_j|$ will always be less than and usually comparable with $\|\mathbf{b}\|_2$.

6: **vl**(**ldvl**,$*$) – **double array**

The first dimension, **ldvl**, of the array **vl** must satisfy

if **jobvl** = 'V', $\mathbf{ldvl} \geq \max(1, \mathbf{n})$;
$\mathbf{ldvl} \geq 1$ otherwise.

The second dimension of the array must be at least $\max(1, \mathbf{n})$ if **jobvl** = 'V', and at least 1 otherwise

If **jobvl** = 'V', the left eigenvectors u_j are stored one after another in the columns of **vl**, in the same order as the corresponding eigenvalues.

If the jth eigenvalue is real, then $u_j = \mathbf{vl}(:,j)$, the jth column of **vl**.

If the jth and $(j+1)$th eigenvalues form a complex conjugate pair, then $u_j = \mathbf{vl}(:,j) + i \times \mathbf{vl}(:,j+1)$ and $u(j+1) = \mathbf{vl}(:,j) - i \times \mathbf{vl}(:,j+1)$. Each eigenvector will be scaled so the largest component has |real part| + |imag. part| = 1.

If **jobvl** = 'N', **vl** is not referenced.

7: **vr**(**ldvr**,$*$) – **double array**

The first dimension, **ldvr**, of the array **vr** must satisfy

if **jobvr** = 'V', $\mathbf{ldvr} \geq \max(1, \mathbf{n})$;
$\mathbf{ldvr} \geq 1$ otherwise.

The second dimension of the array must be at least $\max(1, \mathbf{n})$

If **jobvr** = 'V', the right eigenvectors v_j are stored one after another in the columns of **vr**, in the same order as the corresponding eigenvalues.

If the jth eigenvalue is real, then $v_j = \mathbf{vr}(:,j)$, the jth column of VR.

If the jth and $(j+1)$th eigenvalues form a complex conjugate pair, then $v_j = \mathbf{vr}(:,j) + i \times \mathbf{vr}(:,j+1)$ and $v_{j+1} = \mathbf{vr}(:,j) - i \times \mathbf{vr}(:,j+1)$. Each eigenvector will be scaled so the largest component has |real part| + |imag. part| = 1.

If **jobvr** = 'N', **vr** is not referenced.

8: **info – int32 scalar**

info $= 0$ unless the function detects an error (see Section 6).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

info $= -i$

If **info** $= -i$, parameter i had an illegal value on entry. The parameters are numbered as follows:

1: **jobvl**, 2: **jobvr**, 3: **n**, 4: **a**, 5: **lda**, 6: **b**, 7: **ldb**, 8: **alphar**, 9: **alphai**, 10: **beta**, 11: **vl**, 12: **ldvl**, 13: **vr**, 14: **ldvr**, 15: **work**, 16: **lwork**, 17: **info**.

It is possible that **info** refers to a parameter that is omitted from the MATLAB interface. This usually indicates that an error in one of the other input parameters has caused an incorrect value to be inferred.

info $= 1$ to N

The *QZ* iteration failed. No eigenvectors have been calculated, but **alphar**(j), **alphai**(j), and **beta**(j) should be correct for $j =$ **info** $+ 1, \ldots,$ **n**.

info $= N + 1$

Unexpected error returned from f08xe.

info $= N + 2$

Error returned from f08yk.

7 Accuracy

The computed eigenvalues and eigenvectors are exact for a nearby matrices $(A + E)$ and $(B + F)$, where

$$\|(E, F)\|F = O(\epsilon)\|(A, B)\|F,$$

and ϵ is the ***machine precision***. See Section 4.11 of Anderson *et al.* 1999 for further details.

Note: interpretation of results obtained with the *QZ* algorithm often requires a clear understanding of the effects of small changes in the original data. These effects are reviewed in Wilkinson 1979, in relation to the significance of small values of α_j and β_j. It should be noted that if α_j and β_j are **both** small for any j, it may be that no reliance can be placed on **any** of the computed eigenvalues $\lambda_i = \alpha_i / \beta_i$. You are recommended to study Wilkinson 1979 and, if in difficulty, to seek expert advice on determining the sensitivity of the eigenvalues to perturbations in the data.

8 Further Comments

The total number of floating-point operations is proportional to n^3.

The complex analogue of this function is f08wn.

9 Example

```
jobvl = 'No left vectors';
jobvr = 'Vectors (right)';
a = [3.9, 12.5, -34.5, -0.5;
     4.3, 21.5, -47.5, 7.5;
     4.3, 21.5, -43.5, 3.5;
     4.4, 26, -46, 6];
b = [1, 2, -3, 1;
```

```
     1, 3, -5, 4;
     1, 3, -4, 3;
     1, 3, -4, 4];
[aOut, bOut, alphar, alphai, beta, vl, vr, info] = f08wa(jobvl, jobvr, a,
b)
```

```
aOut =
    3.8009   31.3260  -61.4846  -66.8359
         0    3.3505    7.0744    6.6922
         0   -1.1918    1.4098    4.3790
         0         0         0    4.0000
bOut =
    1.9005   -1.0777   -5.6252   -9.9873
         0    1.1761         0    1.7511
         0         0    0.4474    1.0901
         0         0         0    1.0000
alphar =
    3.8009
    3.0301
    1.5629
    4.0000
alphai =
         0
    4.0401
   -2.0839
         0
beta =
    1.9005
    1.0100
    0.5210
    1.0000
vl =
     0
vr =
    1.0000   -0.4398   -0.5602   -1.0000
    0.0057   -0.0880   -0.1120   -0.0111
    0.0629   -0.1424    0.0031    0.0333
    0.0629   -0.1424    0.0031   -0.1556
info =
           0
```